The probability distribution of the number of equilibria in random replicator-mutator equations for social dilemmas

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Introduction

In many social and biological settings mutation or behavioural exploration is non-negligible (Traulsen et al., 2009; Rand et al., 2013; Zisis et al., 2015). It is also highly relevant in Artificial life systems (Ikegami and Hashimoto, 1995; Bedau, 2003; Sayama, 2011). The replicator-mutator equation (Komarova, 2004) constitutes a fundamental mathematical framework for the modelling, analysis and simulation of such complex systems. It has been used in the study of a variety of problems, including population genetics (Hadeler, 1981), autocatalytic reaction networks (Stadler and Schuster, 1992), language evolution (Nowak et al., 2001) and the evolution of cooperation (Imhof et al., 2005), just to name a few.

Let us assume an infinite population consisting of n types/strategies $S_1, \cdots, S_n$ whose frequencies are, respectively, $x_1, \cdots, x_n$. The reproduction rate of each type, S_i, is determined by its fitness or average payoff, f_i, which is obtained from interacting with other individuals in the population. The interaction of the individuals in the population take place within randomly selected groups of multiple participants. That is, they play and obtain their payoffs from a multi-player game, defined by a payoff matrix. We consider here symmetric games where the payoffs do not depend on the ordering of the players in a group. Due to mutation, individuals spontaneously change from one strategy to another, which is modeled via a row-stochastic matrix (called the mutation matrix), $Q = (q_{ji}), j, i \in \{1, \cdots, n\}$. The entry q_{ji} characterizes the probability that a player of type S_j changes its type or strategy to S_i. The replicator-mutator equation is a set of differential equations describing the evolution of frequencies of different strategies in a population that takes into account both selection and mutation mechanisms, and is given by

$$\dot{x}_i = \sum_{j=1}^{n} x_j f_j(\mathbf{x}) q_{ji} - x_i \bar{f}(\mathbf{x}) =: g_i(\mathbf{x}), \quad i = 1, \ldots, n, \quad (1)$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\bar{f}(\mathbf{x}) = \sum_{i=1}^{n} x_i f_i(\mathbf{x})$ denotes the average fitness of the whole population. The replicator equation is a special case of (1) when the mutation matrix is the identity matrix. In reality, individuals' interactions are often affected by the constantly changing environments making it impossible to assign deterministic payoffs that correctly characterize these interactions. To capture uncertainty, random multi-player multi-strategy games are considered (Duong and Han, 2015; Gokhale and Traulsen, 2010), in which the payoff entries are random variables. In particular, for two-player two-strategy games with a payoff matrix P and a mutation matrix Q given by

$$\begin{array}{c} \\ S_1 \\ S_2 \end{array} \begin{array}{c} \begin{array}{cc} S_1 & S_2 \end{array} \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \end{array}, \quad Q = \begin{array}{c} \\ S_1 \\ S_2 \end{array} \begin{array}{c} \begin{array}{cc} S_1 & S_2 \end{array} \\ \begin{pmatrix} 1-q & q \\ q & 1-q \end{pmatrix} \end{array}$$

where $a_{ij}, i, j \in \{1, 2\}$ is the payoff that a player using strategy S_i obtains when interacting with another player using strategy S_j. The replicator-mutator equation (1) becomes

$$\dot{x} = Ax^3 + Bx^2 + Cx + D, \quad (2)$$

where x is the frequency of the first strategy and $1 - x$ is the frequency of the second one and

$$A = a_{12} + a_{21} - a_{11} - a_{22},$$
$$B = a_{11} - a_{21} - 2(a_{12} - a_{22}) + q(a_{22} + a_{12} - a_{11} - a_{21}),$$
$$C = a_{12} - a_{22} + q(a_{21} - a_{12} - 2a_{22}), \quad D = qa_{22}.$$

Models and Methods

This extended abstract summarizes the main results of recent works in (Duong and Han, 2019, 2021) on the statistics of the number of equilibria in two-player (i.e. pairwise) social dilemma random games. To this end, Duong and Han (2019, 2021) adopt the following parameterized payoff matrix to study the full space of two-player social dilemma games where the first strategy is cooperator and second is defector (Santos et al., 2006; Wang et al., 2015; Szolnoki and Perc, 2019), $a_{11} = 1$; $a_{22} = 0$; $0 \le a_{21} = T \le 2$ and $-1 \le a_{12} = S \le 1$, that covers the following games

(i) Prisoner's Dilemma (PD): $2 \ge T > 1 > 0 > S \ge -1$,

(ii) Snow-Drift (SD) game: $2 \ge T > 1 > S > 0$,

(iii) Stag Hunt (SH) game: $1 > T > 0 > S \ge -1$,

(iv) Harmony (H) game: $1 > T \geq 0, 1 \geq S > 0$.

For these social dilemma games, equation (2) reduces to

$$\Big(T+S-1\Big)x^3+\Big(1-T-2S+q(S-1-T)\Big)x^2+\Big(S+q(T-S)\Big)x = 0. \quad (3)$$

In (Duong and Han, 2019, 2021), the pay-off entries S and T are random variables uniformly distributed in the corresponding interval in each game; for instance, $T \sim U([1, 2])$ and $S \sim U([-1, 0])$ for PD and similarly for other games. The authors established the probability distributions of the number of equilibria for the above social dilemmas. Mathematically, it amounts to find the probabilities that the cubic equation (3) has $k \in \{1, 2, 3\}$ solution(s) in the interval $[0, 1]$. A natural approach is to express the conditions that (3) has $k \in \{1, 2, 3\}$ in terms of the coefficients T and S and then compute the probability for that to happen. The key challenge is that the obtained conditions are complicated. To this goal, Duong and Han (2019, 2021) employ suitable changes of variables, which transform the problem of computing the probabilities to calculating areas, and perform delicate analysis.

Main results

The main results of Duong and Han (2019, 2021) are the following explicit formulas for the probabilities p_i^G that a game $G \in \{SD, H, SH, PD\}$ has $i \in \{1, 2, 3\}$ equilibria:

Suppose that S and T are uniformly distributed in the corresponding intervals as above. Then:

• Snow Drift (SD)

$$p_1 = 0, \quad p_2 = 1, \quad p_3 = 0.$$

• Harmony game (H)

$$p_1 = 0, \quad p_2 = 1, \quad p_3 = 0.$$

• Stag-Hunt game (SH)

$$p_1 = 1 - p_2 - p_3.$$

$$p_2 = \frac{q}{2(1-q)}.$$

$$p_3 = \begin{cases} 1 - \frac{q}{2(1-q)} - \frac{1}{1-2q}\Big[\frac{(3\sqrt{q}+2)^2\sqrt{q}(5q^{3/2}+3q^2-9q-3\sqrt{q}+4)}{12(\sqrt{q}+1)^3}, \\ +\frac{-27q^3-18q^2-32\sqrt{1-2q}q+48q+16\sqrt{1-2q}-16}{12q}\Big], \ 0 < q \leq 4/9, \\ 1 - \frac{q}{2(1-q)} - \frac{8\sqrt{q}(1-2q)^2}{3(1-q)^3}, \quad 4/9 < q < 0.5. \end{cases}$$

• Prisoner's Dilemma (PD)

$$p_1 = 1 - p_2 - p_3.$$

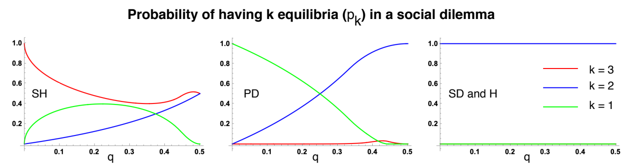
$$p_2 = \begin{cases} \frac{3q}{2(1-q)} & \text{if} \quad 0 < q \leq 1/3, \\ 3 - \frac{1}{2q(1-q)} & \text{if} \quad 1/3 \leq q \leq 1/2. \end{cases}$$

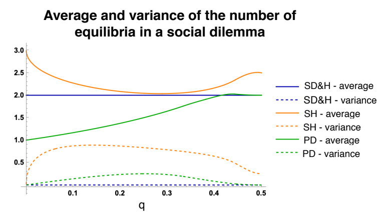
$$p_3 = \begin{cases} \frac{1}{1-2q}\Big[-\frac{2(q^3+3q^2+(4\sqrt{1-2q}-6)q-2\sqrt{1-2q}+2)}{3q} - \frac{1}{2}\frac{q^3}{(1-q)}\Big], \ 0 \leq q \leq \frac{3-\sqrt{5}}{2}, \\ \frac{-16\big(\sqrt{1-2q}-1\big)q^{3/2}+2q^{5/2}+15q^3+\big(8\sqrt{1-2q}-25\big)q^2+q-8\sqrt{1-2q}+2\sqrt{q}\big(8\sqrt{1-2q}-5\big)+5}{6\big(\sqrt{q}-1\big)^3\big(q^{3/2}+q\big)}, \ \frac{3-\sqrt{5}}{2} < q \leq 4/9, \\ \frac{1}{2}\frac{(1-2q)^2}{q(1-q)}, \quad 4/9 < q < 0.5. \end{cases}$$

As a consequence, other statistical quantities such as the mean value, ENoE, and the variance, VarNoE of the number of equilibria can also be derived using the following formulas

$$\mathrm{ENoE} = \sum_{i=1}^{3} i\, p_i, \quad \mathrm{VarNoE} = \sum_{i=1}^{3} p_i(i - \mathrm{ENoE})^2. \quad (4)$$

These quantities are depicted in the Figure below. The results clearly show the influence of the mutation on the number of equilibria in SH-games and PD-games. The probability distributions in these games are much more complicated than in SD-games and H-games and significantly depend on the mutation strength.





Summary and outlook

We have summarized the results of (Duong and Han, 2019, 2021) which provide explicit formulas for the probability distributions of the number of equilibria in term of the mutation probability for random social dilemmas. The analysis is highly important and practical for both social/biological and Artificial life systems, because it is often the case that one might know the nature of an interaction at hand (e.g., a coordination or cooperation dilemma), but it is difficult and/or costly to measure the exact values of the game's payoff matrix (for instance for models with random environments). For future work, it would be interesting and natural question to extend the results to other approaches to studying random social dilemmas such as finite population dynamics and payoff disturbances (Huang and Traulsen, 2010; Szolnoki and Perc, 2019; Amaral and Javarone, 2020a,b), as well as to multi-player (Pacheco et al., 2009; Souza et al., 2009; Luo et al., 2021) and more complex social dilemma games such as the climate change and technological race interactions (Han et al., 2020; Sun et al., 2021).

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